Note

An integral equation representation for overlapping generations in continuous time

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Abstract

This paper develops a method for solving for the dynamic general equilibrium of a deterministic continuous time overlapping generations model with a finite-horizon life-cycle. The model has isoelastic preferences and allows for general assumptions about individual endowments and demographics. Solving for an equilibrium reduces to solving a nonlinear integral equation. In the special case of log utility, the integral equation is linear and global approximations to a solution are easily computed with linear algebra. © 2008 Elsevier Inc. All rights reserved.

1. Introduction

The overlapping generations model is one of the workhorses of modern macroeconomics. This paper outlines a continuous time overlapping generations model where agents live for a given finite interval of time so that genuine life-cycle behavior is possible.[1]

In applied macroeconomics, overlapping generations models are set in discrete time and individuals have many decision periods per lifetime [4,5]. Long-lived models of this kind are attractive because they allow for realistic timing conventions and an intuitive calibration to an-

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[1] Following Yaari [1], Blanchard [2] studies a continuous time overlapping generations model where agents die at an exogenously given exponential rate. This simplifies the analysis considerably but also excludes life-cycle behavior. Weil [3] studies a similar model of overlapping families with infinite horizons. Other related literature is discussed at the end of the paper.

nual or quarterly data. In practice, however, long-lived discrete time models are difficult to work with computationally. The main advantage of the continuous time model outlined below is that it simultaneously permits agents to have many decisions per lifetime but is computationally simple.

The model is presented in Section 2 and allows for isoelastic utility and general assumptions about individual endowments (and demographics). The main result of this paper is given in Section 3 which shows how to represent the equilibrium fixed point problem as a specific, generally nonlinear, integral equation which needs to be solved for an unknown *intertemporal price function*. Section 4 shows for the special case of log utility that this integral equation is linear and solving for an equilibrium involves finding intertemporal prices $p(t)$ satisfying:

$$p(t) = \int_0^\infty p(s)k(t, s)\, ds + f(t)$$

for time $t \in [0, \infty)$. In this equation the integral kernel $k(t, s)$ and forcing process $f(t)$ are known functions of primitives of the economy, including individual endowments and the initial asset distribution. Although this problem generally does not have a closed form solution, there are well known algorithms for finding global approximations to its solution [6–9]. These methods exploit the analogy between the linear integral equation and finite dimensional linear algebra. The case of general isoelastic utility leads to a similar (but nonlinear) integral equation that, as discussed in Section 5, is also computationally tractable—no matter how long-lived individuals are.

2. Model

Consider a deterministic exchange economy populated by an infinity of overlapping generations of agents. Time is continuous and indexed by $t \in [0, \infty)$.

Demographics. The economy is populated by two kinds of agents. First, at each instant of time $t > 0$ a continuum of identical agents are born and live for a finite period of time, $l > 0$. Second, at date $t = 0$ there is a pre-existing mass of agents. Since agents have a lifetime of l, at date $t = 0$ we need to account for all the generations 'born' in the interval $(-l, 0]$. These *transitional* generations play the role of the 'initial old' in the classic two-period overlapping generations model of Diamond [10]. A transitional generation $v \in (-l, 0]$ lives over $t \in [0, v + l]$. Let the set of all generations alive at date $t \geqslant 0$ be $\mathcal{G}(t) := (t - l, t]$. Also, let the set of dates over which generation $v \in (-l, \infty)$ lives be $\mathcal{A}(v) := \{t \geqslant 0 \mid \max[0, v] \leqslant t < v + l\}$. For $v \geqslant 0$ this is just $[v, v + l]$. To simplify notation, let the total population be constant (and normalized to size one).[2]

Endowments. Each instant t generation v is endowed with an exogenous amount of a single non-storable consumption good $y(t, v) \geqslant 0$. No endowment is received if the agent is not alive: $y(t, v) = 0$ for all $(t, v) \notin \mathcal{A}(v) \times \mathcal{G}(t)$. Let $Y(t) := \int_{\mathcal{G}(t)} y(t, v)\, dv > 0$ denote the aggregate endowment at t and let $\varphi(t, v) := y(t, v)/Y(t) \geqslant 0$ denote the density of generation v's endowment.

In addition to physical endowments, transitional generations are endowed with assets—pre-existing claims to consumption. Let $a(0, v)$ denote the net assets of generation v at time $t = 0$.

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[2] Appendix A generalizes the model to allow for aggregate population growth, changes in the relative proportions of young and old, and within-cohort endowment heterogeneity.

For generations $v > 0$ these are zero. For transitional generations $v \in \mathcal{G}(0)$ these may be positive or negative but must net out to zero, $\int_{\mathcal{G}(0)} a(0, v) \, dv = 0$. There is no outside asset.

Preferences and budget constraints. Each individual has preferences over dated consumption goods represented by a time-separable utility function with isoelastic instantaneous utility and exponential discounting:

$$\int_{\mathcal{A}(v)} e^{-\rho(t-v)} u\big[c(t, v)\big] dt, \quad u(c) := \frac{c^{1-\sigma} - 1}{1 - \sigma}, \ \rho \geqslant 0, \ \sigma > 0. \tag{1}$$

For $t \in \mathcal{A}(v)$ the flow constraint facing an individual is:

$$\dot{a}(t, v) = r(t)a(t, v) + y(t, v) - c(t, v) \tag{2}$$

where a dot denotes differentiation with respect to time and with given initial conditions $a(v, v)$ and instantaneous interest rate $r(t)$.

We can integrate the flow constraint to get each individual's intertemporal budget constraint:

$$\int_{\mathcal{A}(v)} p(t)c(t, v) \, dt = \int_{\mathcal{A}(v)} p(t)y(t, v) \, dt + p(v)a(v, v) \tag{3}$$

where $p(t)$ is the intertemporal price of consumption, $\dot{p}(t)/p(t) =: -r(t)$. Using $y(t, v) = 0$ for all $(t, v) \notin \mathcal{A}(v) \times \mathcal{G}(t)$ and $a(v, v) = 0$ for all $v > 0$, the right-hand side of the intertemporal constraint can be written:

$$W(p, v) := \int_0^\infty p(s)y(s, v) \, ds + p(0)a(0, v) \tag{4}$$

where $W(p, v)$ denotes an individual's intertemporal *wealth* as of date $t = 0$ given prices p.

Optimization. Taking prices as given, each individual chooses consumption $c(t, v) \geqslant 0$ to maximize utility (1) subject to their intertemporal constraint (3). This is a concave programming problem over a convex constraint set. The first order condition characterizing consumption is:

$$e^{-\rho(t-v)} c(t, v)^{-\sigma} = \lambda(v)p(t) \tag{5}$$

where $\lambda(v) \geqslant 0$ is the time-invariant Lagrange multiplier on the intertemporal budget constraint of generation v. Differentiating both sides of (5) with respect to t gives the standard consumption Euler equation $\dot{c}(t, v)/c(t, v) = (r(t) - \rho)/\sigma$.

Using the intertemporal constraint, the solution for the Lagrange multiplier is:

$$\lambda_\sigma(p, v) := \left[\int_{\mathcal{A}(v)} e^{-\frac{\rho}{\sigma}(s-v)} p(s)^{\frac{\sigma-1}{\sigma}} \, ds \right]^\sigma W(p, v)^{-\sigma}. \tag{6}$$

Plugging this solution for the multiplier into (5) gives the consumption function:

$$c_\sigma(p, t, v) := \alpha_\sigma(p, t, v) \frac{W(p, v)}{p(t)} \tag{7}$$

where $\alpha_\sigma(p, t, v)$ denotes the expenditure shares:

$$\alpha_\sigma(p, t, v) := \frac{e^{-\frac{\rho}{\sigma}t} p(t)^{\frac{\sigma-1}{\sigma}}}{\int_{\mathcal{A}(v)} e^{-\frac{\rho}{\sigma}s} p(s)^{\frac{\sigma-1}{\sigma}} \, ds}. \tag{8}$$

In the special case of log utility ($\sigma = 1$), we have the familiar property that expenditure shares do not depend on prices p.

Equilibrium. An *equilibrium* is feasible consumption $c \geqslant 0$ and a price $p > 0$ such that (i) taking as given prices, each individual chooses consumption to maximize utility, and (ii) markets clear:

$$\int_{\mathcal{G}(t)} c_\sigma(p, t, v) \, dv = \int_{\mathcal{G}(t)} y(t, v) \, dv =: Y(t). \tag{9}$$

The aggregate endowment $Y(t) > 0$ is exogenous, so we know the right-hand side of (9). We need to find a function p that ensures that markets clear on all dates. That is, we have to solve a nonlinear *integral equation* for p. The market clearing condition gives us an integral equation of the 'first kind,' meaning that the unknown function p enters only inside the integral. In general finding numerical solutions of first kind equations is harder that finding solutions of 'second kind' integral equations where the unknown function also enters outside the integral operator [6,9].[3] In Section 3 below, the properties of the consumption function (7) are used to rewrite (9) as a second kind integral equation.

Because the consumption function is zero degree homogeneous in prices, if p is an equilibrium price, so is ξp for any scalar $\xi > 0$. We can only determine *relative* prices. Let $p(0) = 1$ be the normalization.

Benchmark and the role of compositional effects. Since each individual satisfies their consumption Euler equation, they all have the same consumption growth. But in general this does not mean the growth of the aggregate endowment satisfies a corresponding 'aggregate' Euler equation. In general $\dot{Y}(t)/Y(t) \neq (r(t) - \rho)/\sigma$. To see this, multiply both sides of the Euler equation by $c(t, v)$ and integrate both sides over $\mathcal{G}(t)$ to get:

$$\int_{\mathcal{G}(t)} \dot{c}(t, v) \, dv = \frac{r(t) - \rho}{\sigma} \int_{\mathcal{G}(t)} c(t, v) \, dv = \frac{r(t) - \rho}{\sigma} Y(t).$$

But using Leibniz's rule:

$$\dot{Y}(t) = \frac{d}{dt} \int_{\mathcal{G}(t)} c(t, v) \, dv = c(t, t) - c(t, t - l) + \int_{\mathcal{G}(t)} \dot{c}(t, v) \, dv$$

$$= c(t, t) - c(t, t - l) + \frac{r(t) - \rho}{\sigma} Y(t).$$

Only if the consumption of the very young and very old is the same, $c(t, t) = c(t, t - l)$, does this reduce to the familiar benchmark $\dot{Y}(t)/Y(t) = (r(t) - \rho)/\sigma$ in which case the equilibrium price function is simply $\log[p(t)] = -\rho t - \sigma \int_0^t \gamma(s) \, ds$ with $\gamma(t) := \dot{Y}(t)/Y(t)$ the instantaneous growth rate of the aggregate endowment. See Blanchard [2] and Weil [3] for further discussion.

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[3] If the right-hand side of (9) is smooth then the unknown function p has to be something that *when integrated* gives a smooth answer. But lots of badly behaved functions are smooth after they have been integrated. See [9] for detailed discussion of these *ill-posed* inversion problems.

3. Integral equation representation of equilibrium prices

The main contribution of this paper is a simple functional equation representation of the equilibrium fixed point problem. Intertemporal prices p solve a fixed point problem of the form $p = T_\sigma p$ where T_σ is defined by a nonlinear integral operator.

Proposition 1. *Intertemporal prices p solve the nonlinear integral equation*:

$$p(t) = \int_0^\infty p(s) k_\sigma(p, t, s) \, ds + f_\sigma(p, t) \tag{10}$$

where

$$k_\sigma(p, t, s) := \int_{\mathcal{G}(t)} \alpha_\sigma(p, t, v) \frac{y(s, v)}{Y(t)} \, dv \tag{11}$$

and

$$f_\sigma(p, t) := \int_{\mathcal{G}(t)} \alpha_\sigma(p, t, v) \frac{a(0, v)}{Y(t)} \, dv \tag{12}$$

and where the expenditure shares $\alpha_\sigma(p, t, v)$ are given by Eq. (8).

Proof. Substitute the consumption function $c_\sigma(p, t, v)$ from (7) into the market clearing conditions (9) and multiply both sides by $p(t)$ to get:

$$p(t) Y(t) = \int_{\mathcal{G}(t)} \alpha_\sigma(p, t, v) W(p, v) \, dv. \tag{13}$$

Now substitute in the definition of intertemporal wealth:

$$p(t) Y(t) = \int_{\mathcal{G}(t)} \alpha_\sigma(p, t, v) \left[\int_0^\infty p(s) y(s, v) \, ds + p(0) a(0, v) \right] dv.$$

And change the order of integration for the double integral on the right-hand side of this expression (for this calculation, t is taken as a parameter) to get:

$$p(t) Y(t) = \int_0^\infty \int_{\mathcal{G}(t)} \alpha_\sigma(p, t, v) p(s) y(s, v) \, dv \, ds + p(0) \int_{\mathcal{G}(t)} \alpha_\sigma(p, t, v) a(0, v) \, dv.$$

If we divide both sides by $Y(t) > 0$ and define $k_\sigma(p, t, s)$ and $f_\sigma(p, t)$ as in (11)–(12), then we have the desired representation. □

Eq. (10) is a nonlinear *Urysohn* integral equation [7]. The right-hand side of (10) defines an operator T_σ that takes prices as an argument:

$$(T_\sigma p)(t) := \int_0^\infty k_\sigma(p, t, s) \, ds + f_\sigma(p, t). \tag{14}$$

Equilibrium prices are fixed points of this operator. In practice, approximations to equilibrium prices can be obtained by iterating on T_σ, as outlined below.

4. Solving the model with log utility

In the special case of log utility, the operator T_σ is linear. This is interesting in its own right, but also, because it makes the log case especially easy to solve, helps provide a 'good' initial condition for approximate solutions of the general problem computed by iterating on T_σ.

4.1. Linear integral equation

As $\sigma \to 1$ the expenditure shares no longer depend on p and we have:

$$\lim_{\sigma \to 1} \alpha_\sigma (p, t, v) = \frac{e^{-\rho t}}{\int_{\mathcal{A}(v)} e^{-\rho s} \, ds} =: \alpha(t, v). \tag{15}$$

For generations $v \geqslant 0$ expenditure shares are proportional to $e^{-\rho(t-v)}$ with a constant of proportionality ensuring they integrate to 1 over a lifetime of length l. We now have:

$$\lim_{\sigma \to 1} k_\sigma (p, t, s) = \int_{\mathcal{G}(t)} \alpha(t, v) \frac{y(s, v)}{Y(t)} \, dv =: k(t, s) \tag{16}$$

and

$$\lim_{\sigma \to 1} f_\sigma (p, t) = \int_{\mathcal{G}(t)} \alpha(t, v) \frac{a(0, v)}{Y(t)} \, dv =: f(t) \tag{17}$$

both independent of prices. So the nonlinear Urysohn integral equation (10) becomes:

$$p(t) = \int_0^\infty p(s) k(t, s) \, ds + f(t). \tag{18}$$

This is a linear *Fredholm* equation of the second kind [6–9,11].[4] Equilibrium prices are fixed points $p = Tp$ of the linear operator $T := \lim_{\sigma \to 1} T_\sigma$.

Discussion. Let K denote the linear integral operator implied by the kernel function $k(t, s)$ so that equilibrium prices solve $p = Tp = Kp + f$. Then if the *resolvent operator* $R =: \sum_{i=0}^\infty K^i$ exists, we have:

$$p = (I - K)^{-1} f = Rf = f + KRf.$$

A standard sufficient condition for the resolvent operator to exist is that the operator norm $\|K\| < 1$ [6,11]. Intuitively, this ensures $\sum_{i=0}^\infty K^i$ converges to a well behaved limiting operator.

The properties of equilibrium prices are determined by the known integral kernel function $k(t, s)$ and the known forcing process $f(t)$.

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[4] Many macroeconomists will be familiar with linear Fredholm integral equations of the second kind. A leading example is the Lucas [12] asset pricing model where the unknown function is the price of the single non-storable consumption good in a given state where the state follows an exogenous Markov process with continuous support. See [13] for further discussion.

Integral kernel. The integral kernel $k(t, s)$ encodes information about the (appropriately discounted) relative scarcity of consumption goods at different dates (t, s). Since $y(t, v) = 0$ for all $(t, v) \notin \mathcal{A}(v) \times \mathcal{G}(t)$, the integral kernel can be rewritten:

$$k(t, s) = \begin{cases} 0 & \text{if } s < t - l, \\ k_-(t, s) & \text{if } t - l \leqslant s < t, \\ k_+(t, s) & \text{if } t \leqslant s \leqslant t + l, \\ 0 & \text{if } s > t + l \end{cases} \qquad (19)$$

where

$$k_-(t, s) := \int_{\max\{0, t-l\}}^{s} \alpha(t, v) \frac{y(s, v)}{Y(t)} \, dv, \quad t - l \leqslant s < t \qquad (20)$$

and

$$k_+(t, s) := \int_{\max\{0, s-l\}}^{t} \alpha(t, v) \frac{y(s, v)}{Y(t)} \, dv, \quad t \leqslant s \leqslant t + l. \qquad (21)$$

Forcing process. The know forcing process $f(t)$ inherits its key properties from the distribution of initial assets $a(0, v)$. In particular, because $a(0, v) = 0$ for $v > 0$, we also have $f(t) = 0$ for $t > l$. That is, for $t > l$ there are no more living transitional generations and their *direct* influence on current prices is exactly zero. Of course, the transitional generations continue to have an *indirect* effect. The first round of indirect effects takes place because the forcing process at (say) the point $t = l/2$ directly affects the price at t and indirectly affects all the prices on dates $[0, l/2 + l]$ through the kernel. There is also a whole collection of higher order rounds of indirect effects so the value of the forcing function at any date $t \leqslant l$ matters for the whole price function.

Example: balanced growth. Let the aggregate endowment grow at rate $\gamma \geqslant 0$ so that $Y(t) = e^{\gamma t}$ and let the endowment be uniformly distributed with $\varphi(t, v) = 1/l$ for all $(t, v) \in \mathcal{A}(v) \times \mathcal{G}(t)$. Then for $t > l$ so that we are removed from the influence of the transitional generations:

$$k_-(t, s) = \beta e^{(\gamma + \rho)(s-t)} \left[1 - e^{-\rho(s-t+l)} \right] \qquad (22)$$

and

$$k_+(t, s) = \beta e^{\gamma(s-t)} \left[1 - e^{-\rho(t-s+l)} \right] \qquad (23)$$

where $\beta := 1/[(1 - e^{-\rho l})l]$. Notice that k_- is increasing in s and that k_+ is decreasing in s and $k_-(t, t) = k_+(t, t) = 1/l$. In short, the kernel is a *two-sided smoother* around the point t giving most weight to s close to t (and zero weight to those s that are more than l from the current t). Around t the speed at which the weight given to s falls off depends on the rate of time preference ρ and the growth rate of the aggregate endowment γ. The weights diminish faster when the rate of time preference is higher.

Given that $f(t) = 0$ for $t > l$, the integral equation characterizing equilibrium prices can be written:

$$p(t) = \int_{t-l}^{t+l} p(s) k(t, s) \, ds, \quad t > l. \qquad (24)$$

And with this balanced growth example, from (22)–(23), the integral kernel can be written as a function of $t - s$ only, $k(t, s) = \hat{k}(t - s)$. So (24) provides a simple way of solving for the *steady-state* price path along which $\dot{p}(t)/p(t) = -r$ for some constant interest rate r. We have:

$$1 = \int_{t-l}^{t+1} e^{-r(s-t)} k(t, s) \, ds = \int_{-l}^{+l} e^{-rz} \hat{k}(z) \, dz, \quad t > l \tag{25}$$

which is one equation to be solved for r (there may be several solutions). This interest rate characterizes the limiting equilibrium price path, $p(t) \sim e^{-rt}$ for large t, but since $f(t)$ affects prices at all t, even for this simple example equilibrium prices will not exactly equal e^{-rt} at any t. Also, the steady-state interest rate r does not generally equal the benchmark $\rho + \gamma$ that would obtain with log utility when there are no compositional effects.

One approach to solving the model numerically would be to use local approximation methods based on perturbing the price path $p(t)$ around e^{-rt}. But this neglects the role of transitional generations and makes such solutions unattractive for analyzing both the effects of policy changes along a transition path as well as the full welfare implications of such policy changes. For linear Fredholm integral equations of the second kind, like (18), there are several well known global approximation methods, as outlined below, and so there is no need for local approximations around steady-state.

4.2. Global approximation of $p(t)$ with log utility

The two most common global approximation methods for solving a linear integral equation like (18) are *Nystrom's extension* and the method of approximation by *separable kernels*. Both methods reduce the integral equation problem to a finite dimensional linear algebra problem. Since these methods are well known in the numerical analysis literature [6–9], the discussion here is deliberately terse.

Nystrom's extension. Let s_i, ω_i for $i = 1, \ldots, J$ denote a set of J numerical quadrature nodes and weights that allow us to approximate the integral operator on the right-hand side of (18) by a finite sum:

$$\int_0^\infty p(s) k(t, s) \, ds \approx \sum_{i=1}^J p(s_i) k(t, s_i) \omega_i. \tag{26}$$

Then the linear integral equation can be approximated by the finite system of equations:

$$p_i = \sum_{j=1}^J p_j k_{ij} + f_i, \quad i = 1, \ldots, J \tag{27}$$

where in a slight abuse of notation, $p_i := p(s_i)$, $k_{ij} := k(s_i, s_j)$ and $f_i := f(s_i)$. In matrix notation, $\mathbf{p} = \mathbf{Kp} + \mathbf{f}$. If $\det(\mathbf{I} - \mathbf{K}) \neq 0$ then there is a unique vector, call it $\hat{\mathbf{p}}$, given by the resolvent matrix $\mathbf{R} = (\mathbf{I} - \mathbf{K})^{-1}$ so that $\hat{\mathbf{p}} = \mathbf{Rf} = (\mathbf{I} - \mathbf{K})^{-1}\mathbf{f}$. This gives us approximate equilibrium prices $\hat{p}_i$ at the numerical quadrature nodes s_i, for $i = 1, \ldots, J$. Nystrom's *extension* then gives prices for any $t \in [0, \infty)$ using:

$$p(t) = \sum_{i=1}^J \hat{p}_i k(t, s_i) \omega_i + f(t). \tag{28}$$

Separable kernels. If the integral kernel can be written $k(s, t) = \sum_{i=1}^{J} g_i(s) h_i(t)$ for a finite collection of (known) basis functions $g_i(s)$ and $h_i(t)$ then it is *separable*. In this case, the linear integral equation (18) reduces *exactly* to a finite dimensional linear algebra problem and has the solution:

$$p(t) = \sum_{i=1}^{J} \hat{p}_i h_i(t) + f(t) \tag{29}$$

where $\hat{p}_i$ for $i = 1, \ldots, J$ denote the solutions to the finite system of equations $p_i = \sum_{j=1}^{J} p_j k_{ij} + f_i$ [as in (27)]. And exploiting the separability of the kernel we have exact formulas for the coefficients:

$$k_{ij} := \int_0^\infty h_i(t) g_i(t) \, dt \tag{30}$$

and

$$f_i := \int_0^\infty h_i(t) f(t) \, dt. \tag{31}$$

If the kernel is separable, then (29) gives an *exact* global solution for the equilibrium price function. In general, the kernel will not be separable but can be arbitrarily well approximated as such $k(s, t) \approx \sum_{i=1}^{J} g_i(s) h_i(t)$ for large enough J and appropriate basis functions $g_i(s)$ and $h_i(t)$.

4.3. Numerical examples

The following examples were computed using Nystrom's extension. In each case, the lifetime was set to $l = 75$ years.

Stationary economy. Let the aggregate endowment be constant and the distribution of individual endowments within the current population be uniform, $\varphi(t, v) = 1/l$. The solid line in Fig. 1 is the equilibrium price function when the time discount rate is $\rho = 0.05$. The dashed line is the benchmark $e^{-0.05t}$ which gives the equilibrium prices that would obtain in an analogous representative agent model. The equilibrium real interest rates in this continuous time overlapping generations economy are high. For this example, the initial asset distribution has old agents with positive assets and young agents with negative assets. The top row of Fig. 2 shows the initial asset distribution $a(0, v)$, the corresponding forcing process $f(t)$, and again the equilibrium prices $p(t)$. The bottom row of Fig. 2 shows the reverse case, when the young have assets and the old have liabilities. The forcing process $f(t)$ obtains much of its shape from key properties of the initial asset distribution, so, for example, when the initial old have positive assets then $f(0) > 0$ and $f(t) \to 0^-$ as $t \to l^-$. The reverse is true when the old have initial liabilities. In this example, the equilibrium price function is not very sensitive to properties of the forcing process. The dashed lines show the same objects but with a higher discount rate of $\rho = 0.10$. This has a much greater influence over the whole price function.

Distributional effects. The top panel of Fig. 3 shows the equilibrium price function $p(t)$ for the case of a uniform distribution of individual endowments and for 'hump-shaped' individual endowments that are a quadratic function of age, proportional to $(t - v)(l - t + v)$. The only
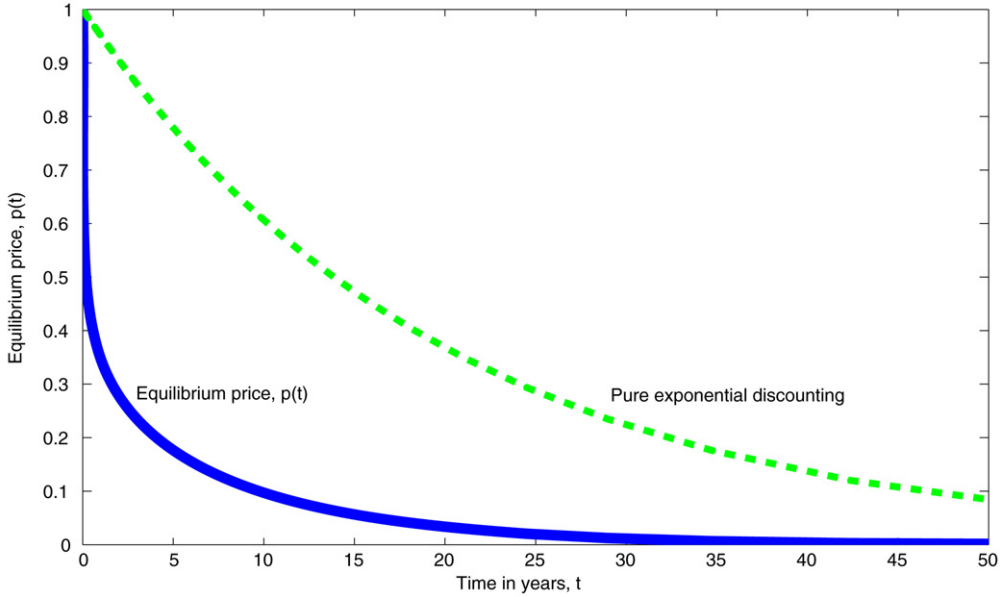
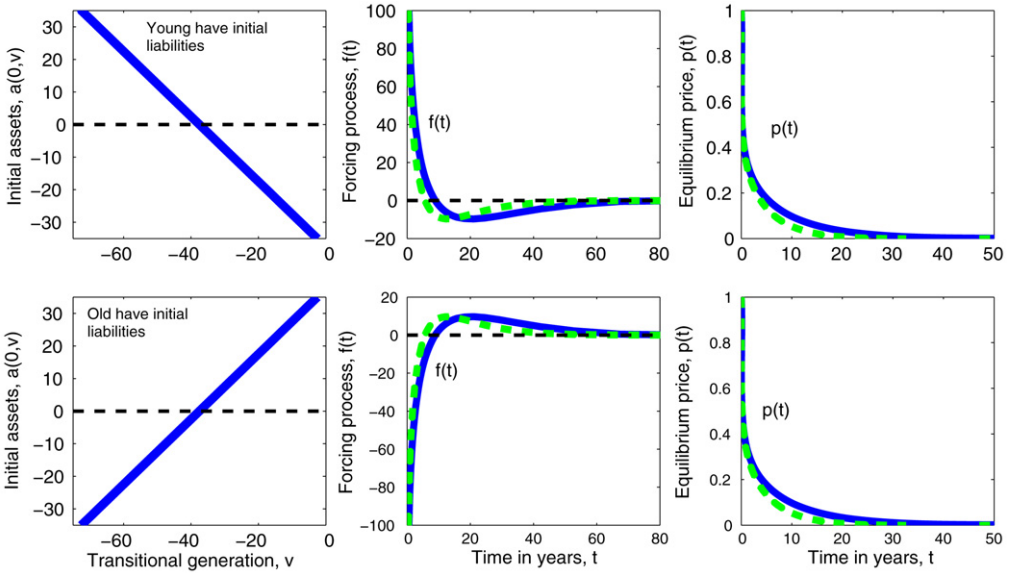Fig. 1. Equilibrium intertemporal prices $p(t)$ versus benchmark $e^{-\rho t}$.

Fig. 2. Sensitivity to initial asset distribution $a(0, v)$. Solid lines have discount rate $\rho = 0.05$, dashed lines have $\rho = 0.10$. Equilibrium prices are much more sensitive to ρ than to the initial distribution.

difference between these economies is the *distribution* of individual endowments. These different patterns of individual endowments lead to quite pronounced differences in equilibrium prices. To get some sense of how big an influence is exerted by the pattern of individual endowments, consider the bottom panel of Fig. 3. This shows the same exercise but with growing endowments
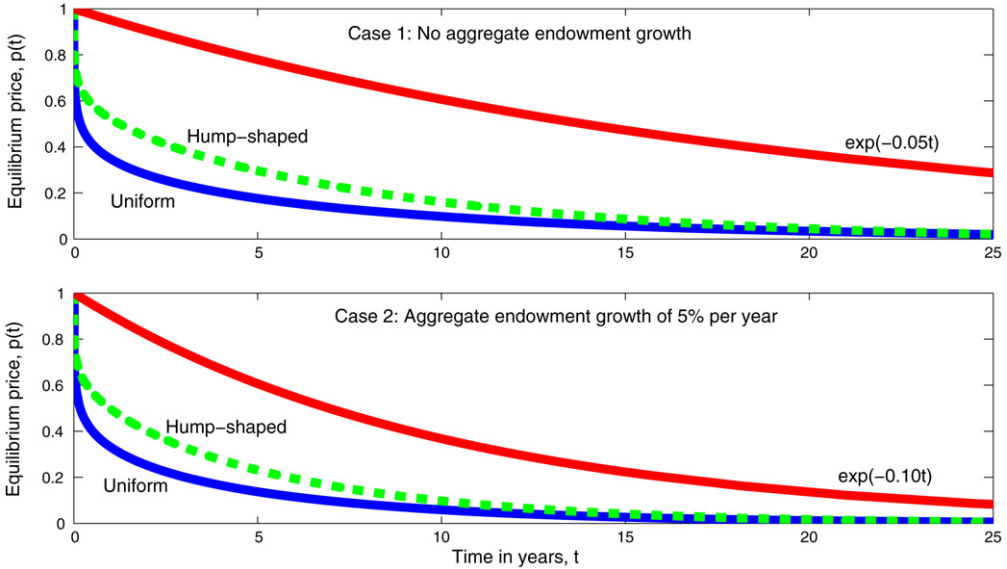
Fig. 3. The change in the distribution of individual endowments (from uniform to 'hump-shaped' in age) has a large effect on $p(t)$. The top panel has discount rate $\rho = 0.05$ and no aggregate endowment growth $\gamma = 0$. The bottom panel has growing aggregate endowment $\gamma = 0.05$.

$Y(t) = e^{0.05t}$. Relative to this increase in the aggregate growth rate, the change in the distribution of individual endowments has a large effect on equilibrium prices.

5. Solving the model with general isoelastic utility

Recall that we are interested in solutions to the fixed point problem $p = T_\sigma p$ where T_σ is the nonlinear operator:

$$(T_\sigma p)(t) = \int_0^\infty p(s) k_\sigma(p, t, s) \, ds + f_\sigma(p, t).$$

Global approximations to the equilibrium price function can be obtained by iterating on an approximating nonlinear operator formed using numerical quadrature. Suppose we have a J-point quadrature rule with nodes t_i and weights ω_i for $t \in [0, \infty)$. Suppose also that we have some J-dimensional *candidate* price vector $\hat{p}_n$ with typical elements $\hat{p}_{n,i}$. For example, this candidate price vector may be the solution from the same model but with log utility.

Now associate with each t_i a set of M quadrature nodes z_{mi} and weights δ_{mi} that give numerical integration over $[t_i - l, t_i)$. Write the approximate expenditure shares given the price function $\hat{p}_n$ as $\hat{\alpha}_n$ with typical elements:

$$\hat{\alpha}_{n,mi} := \frac{e^{-\frac{\rho}{\sigma} t_i} \hat{p}_{n,i}^{\frac{\sigma-1}{\sigma}}}{\sum_{j=1}^J e^{-\frac{\rho}{\sigma} t_j} \hat{p}_{n,j}^{\frac{\sigma-1}{\sigma}} \mathbb{1}\{t_j \in \mathcal{A}(z_{mi})\} \omega_j} \tag{32}$$

for $i = 1, \ldots, J$ and $m = 1, \ldots, M$ (in this expression, $\mathbb{1}$ denotes the indicator function). Similarly compute elements $\hat{k}_{n,ij}$ and $\hat{f}_{n,ij}$ according to:

$$\hat{k}_{n,ij} := \sum_{m=1}^{M} \hat{\alpha}_{n,mi} \frac{y(t_j, z_{mi})}{Y(t_i)} \delta_{mi} \omega_j \tag{33}$$

and

$$\hat{f}_{n,i} := \sum_{m=1}^{M} \hat{\alpha}_{n,mi} \frac{a(0, z_{mi})}{Y(t_i)} \delta_{mi} \tag{34}$$

for $i, j = 1, \ldots, J$. With these formulas in hand, a new estimate of the price vector $\hat{p}_{n+1}$ is computed with typical elements given by the formula:

$$\hat{p}_{n+1,i} = \sum_{j=1}^{J} \hat{p}_{n,j} \hat{k}_{n,ij} + \hat{f}_{n,i} \tag{35}$$

for $i = 1, 2, \ldots, J$.

Now test if the norm $\|\hat{p}_{n+1} - \hat{p}_n\|$ meets some pre-set tolerance criterion. If not, update to (say) $\hat{p}_{n+1}$ and compute new expenditure shares, $\hat{\alpha}_{n+1,mi}$, and new coefficients, $\hat{k}_{n+1,ij}$ and $\hat{f}_{n+1,i}$, using (33)–(34). This procedure is repeated until $\|\hat{p}_{n+1} - \hat{p}_n\|$ is sufficiently small.

6. Related literature

Work on continuous time overlapping generations economies began with Cass and Yaari [14]. They study a model with finite horizons, log utility and physical capital. They derive the possibility of multiple balanced growth equilibria [analogous to multiple solutions r to (25)] and examine the ways in which dynamic efficiency can be addressed in such a setting. Overlapping generations models with continuous time but *infinite horizons* were studied by Blanchard [2] and Weil [3]. Farmer [15] uses a stochastic version of Weil's model for business cycle analysis. The tradition of using long-lived discrete time overlapping generations models for policy analysis is due to Auerbach and Kotlikoff [4]. Ríos-Rull [5] is the standard reference on related long-lived discrete time stochastic overlapping generations economies. Burke [16] tackled the problem of equilibrium existence in a continuous time overlapping generations setting.

A closely related paper is Demichelis and Polemarchakis [17] who discuss a continuous time overlapping generations economy with log utility but no discounting and restrictive assumptions about endowments.[5] D'Albis and Augeraud-Véron have two other closely related papers [18,19]. In [18] they have production and log utility but age-independent wages, i.e., every individual alive at t receives the same competitive wage (there is no heterogeneity in the quantity of efficiency units supplied over the life-cycle). Production exhibits constant returns in capital so the real interest rate and hence intertemporal prices are essentially exogenous. They then derive capital accumulation paths consistent with asset market clearing. They study the problem by recasting the integral equation that emerges naturally from market clearing conditions as a delay-differential equation. In [19] they study an exchange economy with age-independent endowments

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[5] The advantage of these additional assumptions give rise to a linear integral equation of the convolution kind that can essentially be solved in closed form by Fourier transformation.

but general isoelastic utility. The analysis uses local perturbations to a steady-state and ignores transitional generations.

7. Conclusions

This note provides a method for solving for the dynamic general equilibrium of a deterministic continuous time overlapping generations model. The model allows for isoelastic utility and general assumptions about individual endowments (and demographics). The computational task in solving for an equilibrium reduces to solving a specific nonlinear integral equation problem. In the case of log utility the integral equation becomes linear and global approximations can be obtained with finite dimensional linear algebra. The paper outlines a numerical procedure for solving the general integral equation and provides examples.

A continuous time overlapping generations model useful for applied macroeconomics must also be able to handle stochastic environments, capital accumulation and elastic labor supply, amongst other things. Perhaps the framework developed in this paper can be extended along these lines.

Acknowledgments

Appendix A. Variable demographics and within-cohort heterogeneity

The mass of agents of generation $v \in (-l, \infty)$ at date $t \geq 0$ is assumed to be given by a function $n(t, v) \geq 0$. This notation allows for exogenously given changes in the relative proportions of young and old. Let each generation be a continuum $[0, 1]$ of individuals indexed by i with idiosyncratic endowment $y(i, t, v) \geq 0$ at t and aggregate endowment $Y(t) := \int_0^1 \int_{\mathcal{G}(t)} y(i, t, v) n(t, v) \, dv \, di$. Let the initial asset distribution be $a(i, 0, v)$ with $\int_0^1 \int_{\mathcal{G}(0)} a(i, 0, v) n(0, v) \, dv \, di = 0$. Individual consumption is again given by:

$$c_\sigma(p, i, t, v) := \alpha_\sigma(p, t, v) \frac{W(p, i, v)}{p(t)}$$

where $\alpha_\sigma(p, t, v)$ denotes the expenditure shares, the same as in (8), and where individual intertemporal wealth is:

$$W(p, i, v) := \int_0^\infty p(s) y(i, s, v) \, ds + p(0) a(i, 0, v).$$

Market clearing requires $\int_0^1 \int_{\mathcal{G}(t)} c_\sigma(p, i, t, v) n(t, v) \, dv \, di = Y(t)$. This can be rearranged to give the Urysohn integral equation $p(t) = \int_0^\infty p(s) k_\sigma(p, s, t) \, ds + f_\sigma(p, t)$ as in (10) but now:

$$k_\sigma(p, t, s) = \int_0^1 \int_{\mathcal{G}(t)} \alpha_\sigma(p, t, v) \frac{y(i, s, v)}{Y(t)} n(t, v) \, dv \, di$$

and

$$
f_\sigma(p,t) = \int\limits_0^1 \int\limits_{\mathcal{G}(t)} \alpha_\sigma(p,t,v) \frac{a(i,0,v)}{Y(t)} n(t,v)\, dv\, di.
$$

This allows for general endowments and demographics but otherwise the analysis is the same.

Almost immediately this allows the model to be generalized to an economy with deterministic aggregate endowment but *stochastic* idiosyncratic endowments that can be insured away by *dynamically complete* markets (in which case equilibrium individual consumption is deterministic).

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